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Projective Invariance in Cosmological Models and Dark Energy

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! We study theories of gravitation that are based on the Einstein – Hilbert action that are not projectively invariant and can therefore completely determine their connections. We are thus lead to the conclusion that the geometry is necessarily Riemann – Cartan and at least the trace part of a torsion field must be present. We examine the consequence of including these torsion fields in cosmological models. Our results differ from those obtained earlier in the Einstein – Cartan – Sciama – Kibble theory. We also consider a model that includes a series of quadratic torsion terms. This series leads to a potential function that has the effect of "turning on" the cosmological constant. This potential function then acts like dark energy. This model also shows that the torsion field can produce an inflationary period.

KEY WORDS: dark energy; gravitation; Riemann-Cartan; torsion.

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1. INTRODUCTION

The question of the inclusion of torsion in cosmological models was considered in the early years of the development of the Einstein-Cartan-Sciama-Kibble theory, (ECSK), of gravitation, Hehl *et al.* (1976). These models were based on the Einstein-Hilbert action which included a non-symmetric connection and in which the connection was constrained to be metric. Here and throughout this paper metric means that the covariant derivative of the metric tensor with respect to this connection vanishes. The non-symmetric part of the connection is the torsion field. In the ECSK theory, torsion was taken to be the geometrical representation of spin. In fact, the field equations algebraically related the spin field of the matter to the torsion field when the connection is minimally coupled with the matter field. With respect to cosmological models, the torsion field was associated with the spin

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of matter that filled the universe. In some models this matter took the form of a spinning fluid and in others a fermionic matter field. There are numerous articles on this and related topics which may be found in reference (Hehl *et al.*, 1976) and references cited therein, e.g. (Kerlick, 1975; Kuchowicz, 1975; Tafel, 1975).

In this paper we wish to address the question again from a new perspective. First, this perspective recognizes that a theory based on the Einstein-Hilbert action does not completely determine the connection. And secondly, the torsion does not have to be related to spin and could simply be a geometrical quantity characteristic of space-time itself. For example, it has been previously suggested that torsion might be the result of dislocations or defects in space-time that could occur in regions of high curvature (Baker, 1990). This notion would be consistent with the representation of torsion as the failure of the closure of parallelograms formed by the parallel transport of vectors along alternate paths. Finally, we note that our ideas about the universe have matured over time. For example, based on Cosmic Background Radiation (CBR) data, today it is widely believed that the universe is spatially flat. Also we have determined that baryonic matter composes at most 5% of the matter in the universe, while the remaining mass is in the form of dark matter (Freedman and Turner, 2003). Together visible and dark matter compose about 30% of the mass-energy content of the universe with the balance, 70%, beginning in the form of dark energy. Also we have a much better idea of the age of the universe. And finally, we know that the scale factor for the universe in a Robertson–Walker model would have a value at radiation decoupling of one 1100th of its present value. All of this places constraints on viable models of the universe. And in light of these new perspectives it is of some interest to consider torsion fields as participants in the evolution of the universe again. In particular, one might ask if torsion, a property of space-time itself, can be related to either the dark matter or energy. In section two we consider projective invariance and propose a modified action for a gravitational theory. In section three and four we examine the implications of the resulting field equations and in section five we consider the possibility of torsion playing a role as dark energy.

2. PROJECTIVE INVARIANCE

It is well known that Einstein's theory of gravitation, which is based on the Hilbert -Einstein action, is projectively invariant. By projectively invariant we mean that we may add an arbitrary vector field to a connection, i.e. $\Gamma \to \Gamma + B$, and the Ricci scalar, constructed from the new connection, is identical to the Ricci scalar constructed from the old connection. A general projective change on the connection is then given by the following expression; $\Gamma^{\kappa}_{\mu\lambda} \to \Gamma^{\kappa}_{\mu\lambda} + B_{\mu}\delta^{\kappa}_{\lambda}$.

Parallelisms preserving transformations of this type were first considered by Friesecke (1925), Thomas (1926) and Eisenhart (1927). This type of transformation has also been considered by Einstein (1956) in an attempt to construct a unified field theory. When applied to the standard Riemannian connection of Einstein's theory we find, $\{^{\kappa}_{\mu\lambda}\} \to \{^{\kappa}_{\mu\lambda}\} + B_\mu \delta^\kappa_\lambda$, where $\{^{\kappa}_{\mu\lambda}\}$ is the usual Christoffel connection, Schouten (1954). This has the effect of introducing a non-symmetrical connection of the following form, $\Gamma^\kappa_{\mu\lambda} = \{^{\kappa}_{\mu\lambda}\} + B_\mu \delta^\kappa_\lambda$. Because the connection is now non-symmetrical we can identify the associated torsion field as

$$\Gamma^\kappa_{[\mu\lambda]} = S^\kappa_{\mu\lambda} = \frac{1}{2}\left(B_\mu \delta^\kappa_\lambda - B_\lambda \delta^\kappa_\mu\right).$$

In terms of this connection we can determine the vector field B to be $B_\mu = \frac{2}{3}\Gamma^\sigma_{[\mu\sigma]}$, where the symbol $\Gamma^\sigma_{[\mu\sigma]}$ represents the contraction of the torsion tensor over the last two indices. By including this arbitrary vector field we have moved from a Riemannian geometry to a non-Riemannian geometry which is not necessarily metric, i.e. the covariant derivative of the metric tensor, taken with respect to the connection of the geometry, vanishes. Metric geometries that have a non-symmetric connection are Riemann-Cartan. Here the connection then takes the general form $\Gamma^\kappa_{\mu\lambda} = \{^{\kappa}_{\mu\lambda}\} + K^\kappa_{\mu\lambda}$ with $K^\kappa_{\mu\lambda} = S^{\cdot\cdot\kappa}_{\mu\lambda} - S^{\cdot\kappa\cdot}_{\lambda\cdot\mu} + S^{\kappa\cdot\cdot}_{\cdot\mu\lambda}$. In our case we have a contorsion tensor of the form $K^\kappa_{\mu\lambda} = B^\kappa g_{\mu\lambda} - B_\lambda \delta^\kappa_\mu$.

Since we are not necessarily relating this torsion field to the spin of matter, and is therefore not a short range effect as in the case of spin interactions but rather it is a property of space-time, it is important that we consider its implications for the equivalence principle. In both the Einstein and Strong Equivalence Principles (Ciufolini, 1995), it is assumed that we may always find a small region about a test particle where the effects of gravitation cannot be observed. This is usually taken to be a free falling frame, and in this case the motion of the particle would be unaffected by a gravitational field. But particles can execute several types of motion. They may follow geodesies, paths which are auto-parallels or paths that are determined by their equations of motion. In general these possible paths are not the same. First let us consider paths that are determined by the equations of motion.

These equations may be obtained from the field equation or conservation laws and an early example of the procedure is that of Fock (1939). By considering the divergence of the energy momentum tensor for a point particle, or "pole", integrated over an appropriate volume one can find

$$M\left(\frac{d^2x^\alpha}{ds^2} + \left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right) = 0,$$

which also corresponds to a geodesic equation. Later Mathisson (1937) showed that a particle with spin would sense the curvature of the space-time via a curvature spin coupled term. Subsequently Papapetrou (1951); Papapetrou and Corinaldesi (1951), found that, for a "pole-dipole" or spinning particle, the equations of motion are given by:

$$u^{\kappa}\nabla_{\kappa}(mu^{\alpha}) + u_{\theta}(u^{\beta}\nabla_{\beta}S^{\alpha\theta}) + \frac{1}{2}R^{\alpha}_{\rho\kappa\pi}u^{\pi}S^{\rho\kappa} = 0.$$

This is clearly not a geodesic but can be reduced to one if the spin, S, vanishes. For Riemann-Cartan geometries the equations of motion can again be obtained from the energy momentum tensor and following Hehl (1971) we find

$$u^{\kappa}\nabla_{\kappa}(P^{\alpha}) + \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} u^{\gamma}P^{\beta} + K^{\alpha}_{\cdot\gamma\beta}P^{[\beta}u^{\gamma]} = R^{\alpha}_{\rho\kappa\lambda}u^{\lambda}S^{\rho\kappa}.$$

Here P is the momentum of the test particle and u is the four velocity. The third term on the left hand side of this expression involves the contorsion tensor while the right hand side is of the Mathisson form. We note that this expression is functionally similar to the Papapetrou equation. Also, for vanishing spin, the momentum and four velocity are collinear and the equation reduces to that of a geodesic.

In both Riemannian and Riemann-Cartan geometries, when the spin of a test particle vanishes, the equations of motion reduce to the standard geodesic equation. So for non-spinning test particles the EEP and SEP are valid in both geometries.

Because of projective invariance, the vector field B_{μ} cannot be determined from the Einstein-Hilbert action alone. Consequently, part of the torsion field is also not determined by the field equations, Sandberg (1975). Our observations suggest two things. First, we are lead to Riemann-Cartan geometry. Secondly, a new action for a theory that includes this part of the connection is needed.

There are several ways to break projective invariance (Kerlick, 1975). In this paper we will use the means of adding a term which includes the part of the connection which is not now determined by the standard Hilbert-Einstein action in Riemann-Cartan geometry. The modification that we propose is to add a quadratic term in B_{ϕ} which will break the projective invariance of the action. The action then is of the following form

$$S = \int (R(\Gamma) + b_0 B_{\phi}B^{\phi})\sqrt{-g}\,dx. \tag{1}$$

Here b_0 is the constant coefficient for the added non-projective invariance term. The addition of this term breaks the overall projective invariance of the Hilbert-Einstein action and makes the connection fully determinate. At this point we have arrived at an action that is equivalent to an Einstein – Cartan action but with the additional quadratic term in B_ϕ. If we now constrain the geometry to be Riemann-Cartan, we can express the Ricci scalar in terms of the purely Riemann part and the torsion terms, i.e. $R(\Gamma) = R(\{\}) - 6B_\phi B^\phi$ where B_ϕ represents the trace part of the torsion field as noted above. Thus we can expand this action as

$$S = \int (R(\{\}) + (b_0 - 6)B_\phi B^\phi)\sqrt{-g}\, dx. \tag{2}$$

The stress energy tensor that one can obtain from expression (2) by variation with respect to $g^{\mu\lambda}$ is given by

$$G_{\mu\lambda}(\{\}) + (6 - b_0)\left(B_\mu B_\lambda - \frac{1}{2}g_{\mu\lambda}B_\phi B^\phi\right) = 0. \tag{3}$$

We expect that the divergence of this equation should vanish, but this is not the case.

One needs to impose constraints on the B field, i.e. that the divergence is constrained to vanish, and with the addition of a matter term, our total final action is;

$$\begin{aligned} S_{\text{tot}} = \int & (-\kappa^2[R(\{\}) + (b_0 - 6)B_\phi B^\phi] + \phi\nabla_\mu(\{\})B^\mu \\ & + \Im \text{ matter} - \kappa^2\Lambda)\sqrt{-g}dx \end{aligned} \tag{4}$$

We will now turn our attention to the investigation of the field equations that can be obtained from the action S_{tot}.

3. THE FIELD EQUATIONS

The field equations obtained from the variation of S_{tot} with respect to the metric, vector field and multiplier are respectively given by the following:

$$\begin{aligned} &-\kappa^2\left\{G^{\mu\lambda} - \frac{1}{2}g^{\mu\lambda}\Lambda - \Gamma_0\left(B^\mu B^\lambda + \frac{1}{2}g^{\mu\lambda}B_\sigma B^\sigma\right)\right\} + \nabla_\sigma\left(\frac{\phi}{2}B^\sigma\right)g^{\mu\lambda} \\ &-\frac{1}{2}\phi\,(\nabla_\sigma B^\sigma)\,g^{\mu\lambda} + T^{\mu\lambda} = 0 \end{aligned} \tag{5}$$

$$(2\kappa^2\Gamma_0)B_\mu = -\nabla_\mu\phi, \tag{6}$$

$$\nabla_\sigma B^\sigma = 0. \tag{7}$$

Here $\kappa^2 = c^2/8\pi G$ and $\Gamma_0 = b_0 - 6$. We note that we may eliminate the multiplier form the field equations by utilizing Eq. (6) in (5). We then obtain the following expression for the stress energy tensor, Eq. (5):

$$-\kappa^2 \left\{ G^{\mu\lambda} - \frac{1}{2} g^{\mu\lambda}\Lambda - \Gamma_0 \left(B^\mu B^\lambda - \frac{1}{2} g^{\mu\lambda} B_\sigma B^\sigma \right) \right\} + T^{\mu\lambda} = 0$$

From these we will obtain the Friedman equations. We represent matter in the usual manner as a perfect fluid and utilize a spatially flat Robertson-Walker metric, i.e. $ds^2 = c^2\, dt^2 - a^2(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\Phi^2)$, with a time dependent scale factor a. The Friedman equations are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{c^2}{3\kappa^2}\right)\{\rho + \rho_B + \rho_\Lambda\} \tag{8}$$

and

$$2\frac{\ddot{a}}{a} = \left(\frac{-1}{3\kappa^2 c^2}\right)\{\rho + \rho_B + \rho_\Lambda\} - \left(\frac{1}{\kappa^2}\right)\{p + p_B + p_\Lambda\}. \tag{9}$$

A dot over a variable represents differentiation with respect to time.

Here we have identified densities and pressures as $\rho_B = (\Gamma_0\kappa^2 B_4^2)/(2c^2)$, $\rho_\Lambda = \frac{\Lambda}{2}\kappa^2$, $p_B = (\Gamma_0\kappa^2 B_4^2)/2$ and $p_\Lambda = -\frac{\Lambda}{2}\kappa^2 c^2$. We have also assumed that all of the field variables are functions of time only. Consequently, as can be seen from expression 7, B_μ has only one component, i.e. B_4. The field Eq. (7), now in the form, $\dot{B}_4 + (\frac{3\dot{a}}{a})B_4 = 0$, and this determines the vector field to be $B_4 = C_0/a^3$. This indicates that ρ_B is proportional to a^{-6} compared with matter, both visible and dark, which scales as a^{-3}.

In terms of the critical mass density the current Hubble parameter is defined by $H_c^2 = \frac{8\pi G\rho_c}{3}$. We can then write Eq. (8) as

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(H_c^2\right)\left\{\frac{\Omega_m}{a^3} + \frac{\Omega_B}{a^6} + \Omega_\Lambda\right\}. \tag{10}$$

Here the Ω factors are the usual ratios of densities to the critical density. We are now in a position to examine the solution of the Friedman Eq. (10) for this model. A convenient choice is to let $w = a^3$. With this choice Eq. (10) can then be written as

$$\dot{w} = 3H_c\sqrt{\Omega_B + \Omega_m w + \Omega_\Lambda w^2}. \tag{11}$$

We will use this expression to evaluate several possible models.

4. COSMOLOGICAL MODELS WITH TORSION

We find by direct integration that Eq. (11) leads to

$$w = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)\left(\text{Sinh}\left[\frac{3}{2}H_c\sqrt{\Omega_\Lambda}t\right]\right)^2 + \sqrt{\frac{\Omega_B}{\Omega_\Lambda}}\text{Sinh}[3H_c\sqrt{\Omega_\Lambda}t]. \quad (12)$$

This is a general result and the scale factor goes to zero as time goes to zero. This expression is also valid in the case of $\Omega_B = 0$ which would represent the standard model, i.e. 30% matter and 70% dark energy. With Ω_B present, and including the requirement of $\Omega_m + \Omega_\Lambda + \Omega_B = 1$, we find that, as Ω_B becomes increasingly large, the universe becomes younger. Our present perception is that the age of the universe should be about 13.7 billion years. We conclude that to be consistent with this age Ω_B should have a small positive value. This means that with respect to $\Gamma_0 = b_0 - 6$, b_0 would be slightly greater than 6. This leads to the question, what are the consequences of b_0 being less than 6?

It is important to note that for $b_0 \leq 6$ and therefore $\rho_B \leq 0$ we would find also that $\Omega_B \leq 0$, i.e. $\Omega_B = -\|\Omega_B\|$. In this case we can only speak of a minimum value of w. A positive minimum value may be found from the roots of Eq. (11), that is $w_{\min} = (\sqrt{\Omega_m^2 + 4\Omega_\Lambda\|\Omega_B\|} - \Omega_m)/2\Omega_\Lambda$. By once again integrating Eq. (11) we obtain,

$$\begin{aligned} w = \frac{1}{2\Omega_\Lambda}\{&-\Omega_m + (\Omega_m + 2\Omega_\Lambda w_{\min})\text{Cosh}[3H_c\sqrt{\Omega_\Lambda}t] \\ &+ 2\sqrt{\Omega_\Lambda}\sqrt{-\|\Omega_B\| + w_{\min}(\Omega_m + \Omega_\Lambda w_{\min})}\text{Sinh}[3H_c\sqrt{\Omega_\Lambda}t]\}. \quad (13)\end{aligned}$$

By applying the above expression for $w_{\min}$ this expression reduces to

$$w = \frac{1}{2\Omega_\Lambda}\left\{-\Omega_m + \sqrt{\left(\Omega_m^2 + 4\Omega_\Lambda\|\Omega_B\|\right)}\text{Cosh}[3H_c\sqrt{\Omega_\Lambda}t]\right\}. \quad (14)$$

In this case the universe reaches a minimum radius as time goes to zero and does not expand from an initial singularity. We note also that this solution for w is symmetric with respect to the time transformation $t \to -t$. This means that we may speak of this solution as also representing a pre-big bang model. Again for an age of the universe of about 13.7 billion years Ω_B needs to be very small. For small Ω_B, $a_{\min}$ is to a very good approximation

$$a_{\min} = \left[\frac{\|\Omega_B\|}{\Omega_m}\right]^{1/3}. \quad (15)$$

This expression can be used to establish some limits on Ω_B. Since we would expect $a_{\min}$ to be less than the scale factor at radiation decoupling, i.e. less than 1/1100, we conclude that $\|\Omega_B\|$ is less than $10^{-9}\ \Omega_m$. We can see from this analysis that the torsion field determines the minimum size of a universe which has a de Sitter character.

Would the effects of torsion be more pronounced if the cosmological constant was not present? The solution obtained for Eq. (11), with Ω_Λ being set to zero and Ω_B being greater than zero, is given by

$$w = \frac{3}{4}\left(3H_c^2\Omega_m t^2 + 4H_c\sqrt{\Omega_B}t\right). \tag{16}$$

This is a general result and the scale factor goes to zero as time goes to zero. If we specify the matter to be 30% of the critical density and require spatial flatness, then the torsion contribution would be 70% and we would obtain an age for the universe of only about 5 billion years. If we allowed no matter at all the purely torsion universe would be 4.5 billion years old. And finally a flat 100% matter filled universe would have an age of 9 billion years.

For the case that Ω_B is negative we again are faced with a minimum value for the scale factor, $w_{\min} = \|\Omega_B\| / \Omega_m$, and find a general solution of the form

$$w = \frac{1}{4}\left(4w_{\min} + 9H_c^2\Omega_m t^2 + 12H_c t\sqrt{\Omega_m w_{\min} - \|\Omega_B\|}\right).$$

By applying this minimum value we obtain

$$w = \frac{1}{4}\left(\frac{4\,\|\Omega_B\|}{\Omega_m} + 9H_c^2\Omega_m t^2\right). \tag{17}$$

As before, the universe has a minimum radius as time goes to zero and does not expand from an initial singularity. We note also that this is another solution for w which is symmetric with respect to the time transformation $t \to -t$, and so this solution also represents a pre-big bang model. This model is very sensitive to the amount of matter present in the universe. For 100% matter, the minimum scale factor value is zero and the age is 9 billion years. As the mass content decreases the universe gets younger, but the minimum radius, again given by $a_{\min} = [\frac{\|\Omega_B\|}{\Omega_m}]^{1/3}$, increases. For example at 60% matter the scale factor is 0.87, with an age of 6.7 billion years. However, the matter content can not decrease below 50% since a non-real age results. At 50% matter the age of the universe is zero and the minimum scale value is one. This is a completely unrealistic result.

For $a_{\min}$ to be less than the scale factor at radiation decoupling, i.e. less than 1/1100, we conclude that $\|\Omega_B\|$ is less than $10^{-9}\ \Omega_m$. But this is essentially the case of 100% matter, as noted above, which results in a universe that is much too young.

We began this section with an action that included only quadratic terms in the torsion field B_λ, that is $B_\phi B^\phi$. This term broke projective invariance, but higher order terms would also accomplish the same goal. In fact a power series in $B_\phi B^\phi$ would be the most generally allowed combination. We have also seen that our models indicate torsion is significant only for very small scale factors. But, even though it is a small effect, could it contribute to a global factor such as dark energy?

5. TORSION AND DARK ENERGY

Since we are fee to consider a power series in $B_\phi B^\phi$ we have designed a potential that "turns on" the cosmological constant as the torsion field decreases. Consider a variational principle of the following form:

$$S_{\text{tot}} = \int (-\kappa^2[R(\{\}) + \Gamma B_\phi B^\phi] + \phi\nabla_\mu(\{\})B^\mu + \Im\,\text{matter} + \Lambda e^{[-\alpha B_\phi B^\phi]})\sqrt{-g}dx$$

This action is the same as that of Eq. (4) but differs by the inclusion of a potential function, the exponential factor, which contains $B_\phi B^\phi$, along with what would otherwise be the cosmological constant. For high values of $B_\phi B^\phi$ this term is essentially zero but as time progresses it would become larger and the cosmological constant is "turned on".

The resulting field equations following from variation with respect to the metric, the multiplier and the torsion field are respectively:

$$-\kappa^2 G^{\mu\lambda} + \Gamma\kappa^2\left(B^\mu B^\lambda + \frac{1}{2}g^{\mu\lambda}B_\sigma B^\sigma\right) + \frac{1}{2}\nabla_\nu\phi\cdot B^\nu g^{\mu\lambda} + T^{\mu\lambda}$$
$$+\,\alpha\Lambda B^\mu B^\lambda e^{[-\alpha B_\sigma B^\sigma]} - \frac{1}{2}\Lambda g^{\mu\lambda}e^{[-\alpha B_\sigma B^\sigma]} = 0 \tag{18}$$

$$-2\kappa^2\Gamma B_\mu - \nabla_\mu\phi - 2\alpha\Lambda B_\mu e^{[-\alpha B_\sigma B^\sigma]} = 0 \tag{19}$$

$$\nabla_\sigma B^\sigma = 0 \tag{20}$$

Where $\kappa^2 = c^2/8\pi G$.

We may eliminate the multiplier from the stress energy tensor (18), by using Eq. (19) and then it may be rewritten as:

$$-\kappa^2 G^{\mu\lambda} + \Gamma\kappa^2\left(B^\mu B^\lambda - \frac{1}{2}g^{\mu\lambda}B_\sigma B^\sigma\right) + T^{\mu\lambda} + \alpha\Lambda B^\mu B^\lambda e^{[-\alpha B_\sigma B^\sigma]}$$
$$-g^{\mu\lambda}\alpha\Lambda B_\sigma B^\sigma e^{[-\alpha B_\sigma B^\sigma]} - \frac{1}{2}\Lambda g^{\mu\lambda}e^{[-\alpha B_\sigma B^\sigma]} = 0 \tag{21}$$

Also, by forming the curl of the gradient of the multiplier we can eliminate it from Eq. (19) leaving this expression in the following form:

$$(\nabla_\lambda B_\mu - \nabla_\mu B_\lambda)F + B_\mu \nabla_\lambda F - B_\lambda \nabla_\mu F = 0,$$

where $F = -2(\kappa^2\Gamma + \alpha\Lambda e^{[-\alpha B_\sigma B^\sigma]})$.

Finally, we will note that the torsion vector field has spin. This can be seen by considering a Minkowski space-time version of our action, that is;

$$S_{\text{tot}} = \int (L)dx = \int (-\kappa^2[\Gamma B_\phi B^\phi] + \phi\partial_\mu B^\mu + \Lambda e^{[-\alpha B_\phi B^\phi]})dx.$$

The canonical energy-momentum tensor is given by

$$t_\tau^\omega = \frac{\partial L}{\partial B_{\sigma,\omega}} B_{\sigma,\tau} - \delta_\tau^\omega L = \phi n^{\sigma\omega} B_{\sigma,\tau} - \delta_\tau^\omega L.$$

The anti-symmetric portion of this energy-momentum tensor is non-zero, i.e. $t_{[\omega\tau]} = \phi\partial_{[\tau} B_{\omega]}$, and so we can identify the spin from the relationship; $t_{[\omega\tau]} = \partial_\lambda S^\lambda_{\omega\tau}$. Therefore, for the torsion vector field we have a spin tensor given by $S^\lambda_{\omega\tau} = \phi\delta^\lambda_{[\omega} B_{\tau]}$. Having made these observations about the structure of the field equations we will now turn our attention to their cosmological consequences.

Equation (20) for our Robertson-Walker geometry has the solution B_0/a^3. To be consistent with our units we choose the constant B_0 to be $c/\sqrt{\alpha}$.

The associated Friedman equation is given by

$$\left(\frac{\dot a}{a}\right)^2 = \left(\frac{c^2}{3\kappa^2}\right)\{\rho + \rho_B\}. \tag{22}$$

In this expression the densities are identified as follows: $\rho \rightarrow$ The usual matter density which is inversely proportional to the scale factor cubed. We will assume a matter dominated universe so the associated pressure is zero. $\rho_B \rightarrow$ The density associated with the torsion field, and is given by

$$\rho_B = \frac{\Gamma\kappa^2}{2\alpha a^6} - \frac{\Lambda}{2} e^{[-1/a^6]}. \tag{23}$$

The corresponding pressure is

$$p_B = \frac{\Gamma c^2\kappa^2}{2\alpha a^6} + \left\{\frac{1}{a^6} + \frac{1}{2}\right\} c^2 \Lambda e^{[-1/a^6]}. \tag{24}$$

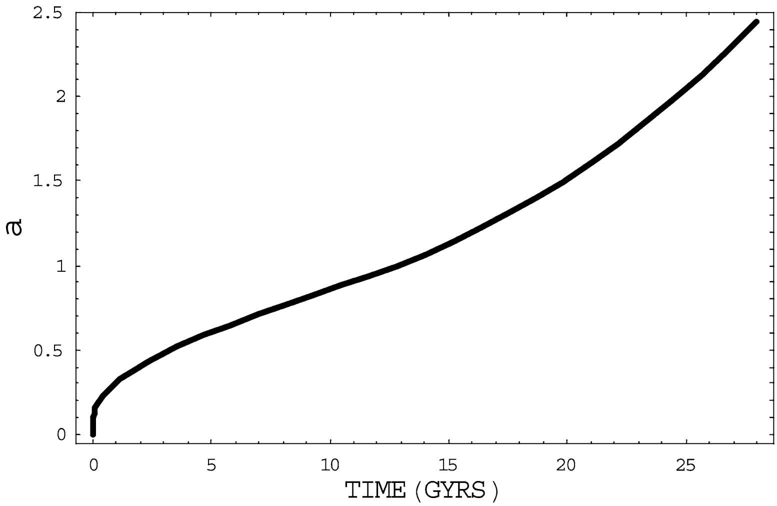

Fig. 1. Scale factor evolution for the model with the torsion potential.

For a model consistent with our present observations Λ will be chosen to be $-\frac{2}{3}\Omega_\Lambda\rho_c e$. With this choice the Friedman equation is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = H_c^2\left\{\frac{\Omega_m}{a^3} + \frac{\Gamma\kappa^2}{2\rho_c\alpha a^6} + e^{(a^6-1)/a^6}(\Omega_\Lambda/3)\right\}. \tag{25}$$

For this model we assume that Ω_m is 0.3 and the value of Ω_Λ, initially unspecified will be found to be about 0.7. Having made the identifications for the various terms, we find by conducting simulations with this model that the best combination which produces a universe with an age of 13.7 billion years is for an Ω_Λ of 0.695, $\alpha = 10^{54}\,\mathrm{m}^2$ and $\Gamma = 50$. Figure 1 shows the evolution of the scale factor for this model.

Figure 2, below, compares this model with the scale factor evolution of the standard model, i.e. Eq. (12) with $\Omega_B = 0$. The agreement between the predictions of the two models is reasonably good. The torsion field model expands at a greater rate earlier in its evolution, reaching the value associated with decoupling, 1/1100, at about 30 years after the Big Bang versus the 454,000 years for the 30% matter and 70% dark energy model.

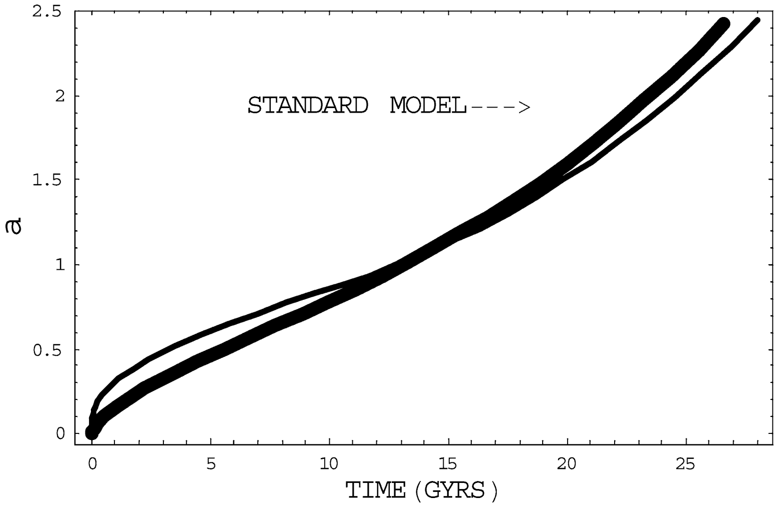

Fig. 2. A comparison of the scale factor evolution for the standard model and the torsion potential model.

The behavior of this model can be understood by realizing that for small scale factor values Eq. (25) basically reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 = H_c^2 \frac{\Gamma\kappa^2}{2\rho_c\alpha a^6}. \tag{26}$$

This has as its solution $a = [3H_c\kappa t(\sqrt{\Gamma/2\alpha\rho_c})]^{1/3}$.

Compared with the standard model at early times, i.e. $a = [(\frac{\Omega_m}{\Omega_\Lambda})(\mathrm{Sinh}[\frac{3}{2}H_c\sqrt{\Omega_\Lambda}t])^2]^{1/3}$, we see that the standard model expands at a much slower rate, which is on the order of 10^{-17} times less than that of the torsion model at 10^{-32} s. So we see that the model with torsion has an inflationary aspect. At about 8 billion years the last term in Eq. (25) suddenly increases from a near zero value as the torsion potential becomes significant.

The effect of this term can be seen in the pressure versus density and the deceleration evolution, Figs. 3 and 4. If we examine the pressure versus density evolution for the B_λ field, i.e. $p_B/(\rho_{\mathbf{B}}c^2)$, we find that it becomes increasingly negative with time, Fig. 3.

In this particular model there is a rapid transition from 1 to -3.3 that occurs between 8 and 11 billion years, afterwards it gradually returns to a value of -1. For the deceleration parameter there is a transition form a positive value to a negative value as well.

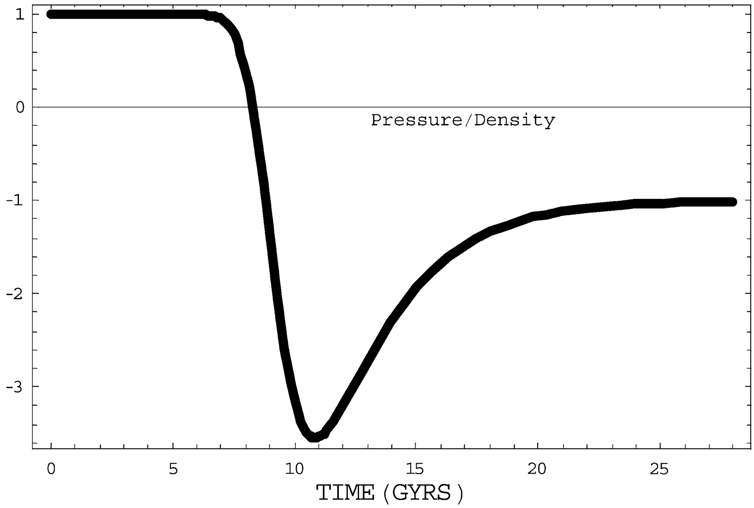

Fig. 3. The ratio or pressure to density as a function of time in billions of years.

When this transition occurs is determined by the "turning" on of the torsion potential which is controlled by the $e^{(a^6-1)/a^6}$ factor, see Eq. (25). Figure 4 shows the change of the deceleration parameter q and the pressure to density ratio of the B_λ field along with the $e^{(a^6-1)/a^6}$ factor.

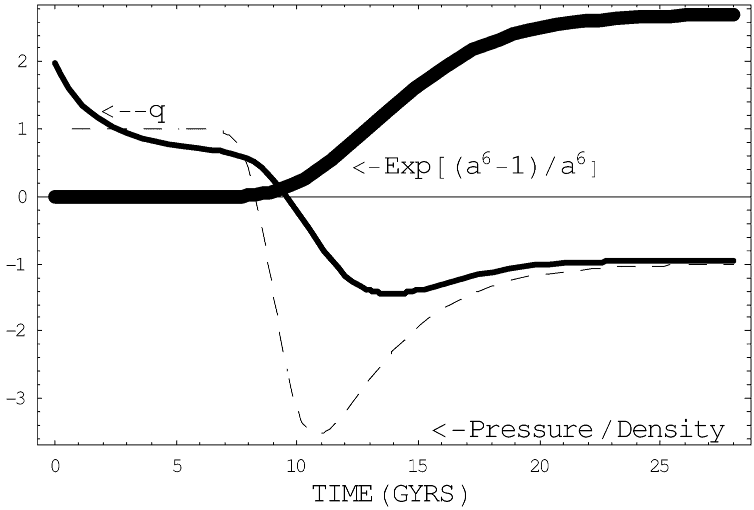

Fig. 4. Plot of the deceleration parameter, q, pressure to density ratio and $e^{(a^6-1)/a^6}$.

It should be noted for this model that, with respect to the deceleration parameter q, the universe undergoes a period of deceleration for about 8 billion years followed by acceleration which then settles down to a q value of -1. The change of these parameters is due to the specific form of the potential that we are using. The details could possibly be altered by refinements of the functional form of the potential. The important feature is that, as in the case of the standard model, the deceleration goes from a positive to a negative value as the universe evolves. When this transition of q and the pressure to density ratio to more negative values occur can be adjusted by varying the α parameter. Smaller values than the nominal value result in transitions at a much earlier time in the evolution of the universe, and a much younger universe. So the nominal value that we have found here is the smallest reasonable value. It is worth noting that for $\alpha = 10^{54}\mathrm{m}^2$ the current magnitude of the torsion field ($\sqrt{B_\phi B^\phi}$) is $1/\sqrt{\alpha}$ which is very small. This value is consistent with the belief that any torsion field present would have a small value.

6. TORSION AS DARK ENERGY VERSUS K-ESSENCE

Because the torsion field that we are considering has only one component, it might be treated as being determined by the gradient of a scalar field. In that case, the model presented here could be viewed as a sub case of a quintessence (Caldwell *et al.*, 1998; Ferreira and Joyce, 1997; Frieman *et al.*, 1995; Ratra and Peebles, 1988) or k-essence (Armendariz-Picon *et al.*, 1999; Chiba, 2002; Armendariz-Picon *et al.*, 2000; Chiba *et al.*, 2000; Chimento and Feinstein, 2004; Malquarti *et al.*, 2003; Scherrer, 2004) theory. Because of this, it is worth considering a scalar field version, similar to those considered in k-essence models, based on the same general action as we chose for the torsion field. Therefore, consider a model with the following action:

$$S_{\mathrm{tot}} = \int (-\kappa^2 R(\{\}) + \Im\,\mathrm{matter} + Af\Phi_{,\mu}\Phi^{,\mu} + De^{[-\beta f \Phi_{,\mu}\Phi^{,\mu}]})\sqrt{-g}dx \quad (27)$$

Here we have included a function, f, that is dependent on the quantity $\Phi_{,\mu}\Phi^{,\mu}$ which we shall designate as X for brevity, i.e. $f(X)$. This function is in general a polynomial in X and we shall assume that is of the form, $\sum_n C_n X^n$. The following field equations that result from variation with respect to the metric and scalar field are respectively:

$$\begin{aligned}-\kappa^2 G^{\mu\lambda} + T^{\mu\lambda} - Af\frac{g^{\mu\lambda}}{2}\Phi_{,\rho}\Phi^{,\rho} - \frac{g^{\mu\lambda}}{2}De^{[-\beta f\Phi_{,\rho}\Phi^{,\rho}]}\\ + g^{\mu\tau}g^{\lambda\kappa}\Phi_{,\tau}\Phi_{,\kappa}\left(A - \beta De^{[-\beta f\Phi_{,\rho}\Phi^{,\rho}]}\right)\left(f + \frac{\partial f}{\partial X}X\right) = 0 \qquad (28)\end{aligned}$$

and

$$\partial_\lambda\left[\sqrt{-g}(g^{\mu\lambda}\Phi_{,\mu}\big\{-A+\beta De^{[-\beta f\Phi_{,\rho}\Phi^{,\rho}]}\big\}\left\{f+\frac{\partial f}{\partial X}X\right\}\right]=0. \qquad (29)$$

We can identify the corresponding density and pressure associated with the scalar field as

$$\rho_\Phi=\frac{A}{2}f\frac{\dot{\Phi}^2}{c^2}-\left[\frac{\beta f\dot{\Phi}^2}{c^2}+\frac{1}{2}\right]De^{[-\beta\dot{\Phi}^2/c^2]}+(A-\beta De^{-\beta f\dot{\Phi}^2/c^2})\frac{\partial f}{\partial X}\frac{\dot{\Phi}^4}{c^4} \quad (30)$$

and,

$$\frac{p_\Phi}{c^2}=\frac{A}{2}f\frac{\dot{\Phi}^2}{c^2}+\frac{D}{2}e^{[-\beta f\dot{\Phi}^2/c^2]}. \qquad (31)$$

Making the usual assumption that the fields are time dependent only we can show that expression (29) can be simplified as

$$\dot{\Phi}\big(-A+\beta De^{[-\beta\dot{\Phi}^2/c^2]}\big)\left(f+\frac{\partial f}{\partial X}\frac{\dot{\Phi}^2}{c^2}\right)=\frac{w_0}{a^3} \qquad (32)$$

Here w_0 represents the current values for the scalar field derivatives, i.e. $\dot{\Phi}_0(-A+\beta De^{[-\beta\dot{\Phi}_0^2/c^2]})(f+\frac{\partial f}{\partial X}\frac{\dot{\Phi}_0^2}{c^2})$. Equation (32) is in general a transcendental equation and does not have a simple closed from solution. Further the details of $f(X)$ need to be specified.

As a starting point let us assume that $f(X)$ is a constant. Since we expect that early in the evolution of the universe $\dot{\Phi}$ should have a large magnitude the expression (32) tells us that the following relationship should be approximately true $\dot{\Phi}=\frac{w_0}{(-A)a^3}$. Formally, this is in agreement with the vector torsion field case. However, late in the evolution, as a becomes larger, we expect $\dot{\Phi}$ to approach a zero value. To a very good approximation the field equation is then given by

$$\dot{\Phi}(-A+\beta D(1-\beta\dot{\Phi}^2/c^2))=\frac{w_0}{a^3}. \qquad (33)$$

This can be solved for $\dot{\Phi}$, which gives us an expression that assumes a limiting value of zero as a becomes infinitely large only if $A=\beta D$. Imposing this requirement leads to a rather simple solution of the form

$$\dot{\Phi}=\frac{n_0}{(-A)a}, \qquad (34)$$

where we have chosen A such that $\beta w_0^2 = A^2c^2$ and $n_o^2 = \frac{c^2}{\beta}$.

This differs from the torsion field case which is always inversely proportional to a^3.

Now let us assume that $f(X)$ is no longer a constant but takes the form of a polynomial. In this case the quantity $(f + \frac{\partial f}{\partial X}\frac{\Phi^2}{c^2})$ can be expressed as $\sum_{j=0}^{N}(j+1)C_jX^j$.

Again, if we assume that $\dot{\Phi}$ should have a large magnitude early in the evolution of the universe this series will be dominated by its highest order term, N, of the polynomial. In this case Eq. (32) can be expressed as $\dot{\Phi}A(C_N(N+1)\frac{\dot{\Phi}^{2N}}{c^{2N}}) = \frac{w_0}{a^3}$ which leads to $\dot{\Phi} = \frac{w_0'}{a^{3/(2N+1)}}$. We note that as the polynomial order increases $\dot{\Phi}$ approaches a constant value. Again, this differs from the torsion field case which is always inversely proportional to a^3.

Finally for small values $\dot{\Phi}$ only the lowest order terms of a polynomial for $f(X)$ would apply and we would find that (32) can be approximated as $\dot{\Phi}\{-A + \beta D - \beta^2 Df\dot{\Phi}^2\}\{C_0 + C_1\dot{\Phi}^2\} = \frac{w_0}{a^3}$, so that to third order in $\dot{\Phi}$ we find $\dot{\Phi} = \frac{w_0''}{a}$.

Consequently the torsion vector field model does not, in general, yield the same behavior as the scalar field cases posed either by k-essence or in a simplified scalar model, and so our model is distinct from those models that employ scalar fields. We further note that none of these scalar fields have spin, which again distinguishes them from the torsion field case.

7. CONCLUSIONS

Our work has shown that by breaking the projective invariance of the action for Einstein's theory of gravitation we are able to introduce torsion, specifically the trace part of the more general torsion tensor, in a natural way which is unrelated to the spin of a matter field. In this regard torsion is distinct from that found in the ECSK. We began with a simple quadratic combination of the torsion vector field which was added to the Hilbert-Einstein action. Cosmological models based on this now non-projectively invariant action were then considered. These models showed that torsion would be significant in the early universe, but the ages of the universe predicted by these models would not in general be consistent with our current understanding. Also, by choosing an appropriate value for the coupling constant of the quadratic torsion term, the initial singularity could be avoided. Although this effect is similar to that found in the ECSK theory, the causes are not the same.

Finally, we considered a model that includes a series of quadratic torsion terms that break projective invariance. This series, which is an exponential function, leads to a potential function that has the effect of "turning on" the cosmological constant. This potential function then acts like dark energy as the torsion field diminishes with the expansion of the universe. This model also shows that although the magnitude of the torsion field in the early universe is small it can produce an inflationary period. The evolution of the scale factor and age of the universe as predicted by this new model are consistent with our current understanding of the universe. It also predicts radiation decoupling earlier than that by the standard model, i.e. 30 years versus 450,000 years. By adding the torsion field in concert with dark energy, this model has given a geometrical origin to this enigmatic phenomenon.

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